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Octonionic Yang–Mills instanton on quaternionic line bundle of *Spin*(7) holonomy

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Abstract

The total space of the spinor bundle on the four-dimensional sphere S^4 is a quaternionic line bundle that admits a metric of *Spin*(7) holonomy. We consider octonionic Yang–Mills instanton on this eight-dimensional gravitational instanton. This is a higher dimensional generalization of (anti-) self-dual instanton on the Eguchi-Hanson space. We propose an ansatz for *Spin*(7) Yang–Mills field and derive a system of non-linear ordinary differential equations. The solutions are classified according to the asymptotic behavior at infinity. We give a complete solution when the gauge group is reduced to a product of *SU*(2) subalgebras in *Spin*(7). The existence of more general *Spin*(7) valued solutions can be seen by making an asymptotic expansion.

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1. Introduction

Instantons and soliton solutions have played a prominent role in our understanding of non-perturbative dynamics and dualities in gauge field theories and string theory. It has been observed that fundamental examples of topological solutions are associated with the four Hopf fibrations of spheres, which in turn are related to the division algebras of real numbers **R**, complex numbers **C**, quaternions **H** and octonions **O** [1–3]. The kink solution in $1+1$ dimensions, the Dirac monopole in three dimensions and the *SU*(2) Yang–Mills

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instanton in four dimensions correspond to the first three algebras. In this paper we will bring into focus eight-dimensional instantons that corresponds to the octonions.

If the theory is promoted to a supersymmetric (SUSY) theory, these topological solutions obtain a new feature. That is, they are characterized as BPS states that preserve a fraction of SUSY. Roughly speaking a first order soliton equation is a "square root" of the equation of motion for bosons and hence appears in the SUSY transformations of fermions. Solutions that make the SUSY variation of fermions vanishing give purely bosonic configurations which preserve (or break) some portions of supersymmetry. It is interesting that there is also a relation between the four division algebras and the existence of supersymmetric pure Yang–Mills theory and superstring theory in $d = 3, 4, 6, 10$ [4,5]. Thus we see there are amusing links among instanton, supersymmetry and the division algebra. From this viewpoint octonionic instanton that will be featured in the following is related to SUSY Yang–Mills and superstring theory in 10 dimensions.

Topological quantum field theories enter naturally in these connections. Especially, topological Yang–Mills theories in two and four dimensions are associated with the complex numbers and the quaternions, respectively. Furthermore, one can construct an eight- dimensional cohomological Yang–Mills theory based the octonionic instanton equation [6–8], though it makes sense only on a manifold of restricted holonomy. It is promising that this BRST cohomological theory probes the moduli space of the octonionic instanton equation. But how this is achieved actually depends on our finding an appropriate compactification of the moduli space. In the case of four-dimensional instantons we need the point-like (ideal) instantons to compactify the moduli space. In [9] it has been argued that a natural higher dimensional analogue of ideal (point-like) instanton is that lives on the normal bundle over a supersymmetric cycle (or a calibrated submanifold) that has codimension four. This argument gives a good motivation for looking at higher dimensional instanton on the $\mathbf{R}^4$ bundle.

Fortunately an eight-dimensional metric that is prepared for this purpose has been provided in [10–12]. It is a metric of $Spin(7)$ holonomy on the $\mathbf{R}^4$ bundle over S^4, which is an example of a quaternionic line bundle over a quaternionic Kähler manifold. In this paper we consider the octonionic Yang–Mills instanton on this $Spin(7)$ holonomy manifold. As will be shown in Section 2 this metric is a natural eight-dimensional generalization of the Eguchi-Hanson metric. The geometry of quaternions replaces the role of complex numbers in the Eguchi-Hanson space whose global structure is the canonical line bundle over the complex projective space $P_1(\mathbf{C}) \cong S^2$. Thus we can say that our instanton is a higher dimensional generalization of (anti-)self-dual instanton on the ALE space. It is known that the spinor bundle over a four-dimensional spin manifold has $Spin(7)$ holonomy in general. The quaterninonic line bundle in this paper is identified with the spinor bundle over S^4 (see Appendix A).

The paper is organized as follows; in Section 2 we introduce the metric of Gibbons–Page–Pope and set up notations that are necessary in the following sections. The metric of $Spin(7)$ holonomy is a solution to the octonionic self-duality for spin connections. We show that the same condition is obtained as a flow equation in a SUSY quantum mechanics of weight functions appearing in our ansatz of the metric. In a course of explaining the implication of the octonionic self-duality, some algebraic properties of *Spin*(7) as a subalgebra of *SO*(8) are reviewed briefly. We propose our ansatz for *Spin*(7) Yang–Mills field in Section 3, and write down the octonionic Yang–Mills instanton equation. To solve the instanton equation we first make a classification of solutions according to the asymptotic behavior at infinity and find six classes. There are reduced solutions in the sense that the actual gauge group is reduced to a direct product of *SU*(2) factors in *Spin*(7). For these class of solutions we present almost complete answer in Section 4. Unfortunately for other solutions we are not able to find solutions in analytically closed form. In Section 5 we perform asymptotic expansion to see the existence of general solutions. The final section is devoted to discussion. We point out the relation to seven-dimensional Chern–Simons theory. We also make a remark on the energy–momentum tensor of higher dimensional Yang–Mills instantons.

2. Gravitational instanton in eight dimensions

We first derive a metric of *Spin*(7) holonomy from the viewpoint of supersymmetric quantum mechanics. This metric was originally obtained by Gibbons–Page–Pope [10,11] and further discussed in [12]. (See also [13] for more intrinsic definition of the metric.) We take the following ansatz for a metric on the $\mathbf{R}^4$ bundle over S^4:

$$d\hat{s}^2 = f^2\, dr^2 + g^2\, ds^2 + h^2(\sigma_i - A_i)^2, \tag{2.1}$$

where

$$ds^2 = d\mu^2 + \tfrac{1}{4}\sin^2\mu \cdot \Sigma_i^2 \tag{2.2}$$

is the standard metric on the base space S^4. We assume that f, g and h are functions of the radial co-ordinate r. Σ_i and σ_i are left-invariant one-form of *SU*(2) manifold:

$$d\Sigma_i = -\tfrac{1}{2}\epsilon_{ijk}\Sigma_j \wedge \Sigma_k, \qquad d\sigma_i = -\tfrac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k. \tag{2.3}$$

Note that $SU(2) \cong Sp(1)$ is the space of quaternions with unit norm. Finally A_i represents the basic *SU*(2) instanton on S^4:

$$A_i = \cos^2\frac{\mu}{2} \cdot \Sigma_i. \tag{2.4}$$

The vielbein (orthonormal frame) of the above metric is

$$\begin{aligned} &e^i = \tfrac{1}{2}g(r)\sin\mu \cdot \Sigma_i, \qquad e^{\hat{i}} = h(r)(\sigma_i - A_i), \qquad e^7 = f(r)\, dr, \\ &e^8 = g(r)\, d\mu, \end{aligned} \tag{2.5}$$

where $i = 1, 2, 3$ and $\hat{i} = 4, 5, 6 = \hat{1}, \hat{2}, \hat{3}$. The indices (1, 2, 3, 8) are for the base space S^4 and (4, 5, 6, 7) are those for the fiber. This metric is a special case of more general class of metric on the quaternionic line bundle over a quaternionic Kähler manifold. Note that $S^4 = P_1(\mathbf{H})$ is not a Kähler but a quaternionic Kähler manifold. In fact any four-dimensional manifold is quaternionic Kähler. The Eguchi-Hanson metric is a four-dimensional gravitational instanton on a (complex) line bundle over the complex projective space $P_1(\mathbf{C})$.

More precisely the global topology of the Eguchi-Hanson space is the co-tangent bundle $T^*(P_1(\mathbf{C}))$ which coincides with the canonical bundle. Hence, the above metric is a higher dimensional generalization of the Eguchi-Hanson metric obtained by simply replacing the complex numbers with the quaternions (see also Appendix A).

It is straightforward to compute the Ricci tensor of the metric (2.1), which is found to be diagonal:

$$Ric_{ij} = \delta_{ij}\left\{\frac{3}{g^2}\left(1-\frac{1}{2}\frac{h^2}{g^2}\right) - 4K^2 - \frac{K'}{f} - 3KL\right\},$$
$$Ric_{\hat{i}\hat{j}} = \delta_{ij}\left\{\frac{h^2}{g^4} + \frac{1}{2h^2} - \frac{L'}{f} - 4KL - 3L^2\right\},$$
$$Ric_{77} = -4\frac{K'}{f} - 4K^2 - \frac{3}{f}L' - 3L^2, \qquad Ric_{88} = Ric_{ii}, \tag{2.6}$$

where

$$K = \frac{g'}{fg}, \qquad L = \frac{h'}{fh}, \tag{2.7}$$

and $'$ denotes the differentiation. The volume form of the manifold is given by

$$e^1 \wedge e^2 \wedge \cdots \wedge e^8 = fg^4h^3\,\mathrm{d}r \wedge \mathrm{d}\Omega_{S^7}, \tag{2.8}$$

where $\mathrm{d}\Omega_{S^7}$ is the volume form of the sphere S^7. We obtain the Einstein–Hilbert action:

$$\int R\sqrt{g}\,\mathrm{d}^8x = vol_{S^7}\cdot(T+V),$$

where

$$T = \int f^{-1}\,\mathrm{d}r\left(12(g')^2g^2h^3 + 6(h')^2g^4h + 24(g'h')g^3h^2\right), \tag{2.9}$$

$$V = \int f\,\mathrm{d}r\left(12g^2h^3 - \frac{3}{2}g^4h - 3h^5\right). \tag{2.10}$$

Note that we are going to regard the radial co-ordinate r as a "time" and hence the relative sign of the kinetic term and the potential term in the action is changed due to the Euclidean time. With this interpretation the function $f(r)$ is a gauge freedom of time reparametrization. In fact we would be able to impose a gauge fixing condition $f = 1$. Hence the physical variables are $g = g(r)$ and $h = h(r)$. The (sigma model) metric determined from the kinetic term T is

$$G_{gg} = 24f^{-1}g^2h^3, \qquad G_{gh} = G_{hg} = 24f^{-1}g^3h^2, \qquad G_{hh} = 12f^{-1}g^4h. \tag{2.11}$$

Now we are ready to make a crucial observation that the present model can be identified as a bosonic part of supersymmetric quantum mechanics (0+1 dimensional SUSY sigma model). The point is that introducing the superpotential

$$W = 3g^4h^2 + 6g^2h^4, \tag{2.12}$$

we can write the potential term in the form

$$V = \frac{1}{2} G^{ij} \frac{\partial W}{\partial q^i} \frac{\partial W}{\partial q^j}, \tag{2.13}$$

where $(q^1, q^2) = (g, h)$ and G^{ij} is the inverse of the metric. The equation of motion from the Hamiltonian [1] $H = T - V$ of SUSY quantum mechanics

$$\frac{\mathrm{d}}{\mathrm{d}r} q^i = -G^{ij} \frac{\partial W}{\partial q^j} \tag{2.14}$$

gives the following flow equations:

$$K = \frac{g'}{fg} = -\frac{3}{2} \frac{h}{g^2}, \qquad L = \frac{h'}{fh} = -\frac{1}{2h} + \frac{h}{g^2}. \tag{2.15}$$

If we introduced fermionic variables that are SUSY partners to $(q^1, q^2) = (g, h)$, these equations would be equivalent to the condition that SUSY variations of the fermionic co-ordinates vanish and hence determine a purely bosonic configuration that is invariant under supersymmetry. We observe that (2.15) coincides with the condition of the octonionic self-duality of the Riemann curvature, which shows the BPS nature of the octonionic self-duality. Using (2.6) and (2.15), we can see that a solution to the above equation gives a Ricci-flat metric. It is known that the same structure arises for the four-dimensional hyperKähler metrics that depends only on a radial co-ordinate regarded as an Euclidean time in SUSY quantum mechanics. We note that what we have shown for the *Spin*(7) holonomy is also valid for a seven-dimensional metric with G_2 holonomy in [11]. It is an interesting problem to work out a similar relation to SUSY quantum mechanics for other examples of the metric with special holonomy.

In [12] (2.15) are derived from the self-duality on the spin connection

$$\omega_{ab} = \tfrac{1}{2} \Psi_{abcd} \omega^{cd}, \tag{2.16}$$

where a totally anti-symmetric tensor Ψ_{abcd} is defined in Appendix B in terms of the structure constants of octonions. Imposed with a gauge condition on $g(r)$ instead of $f(r)$, (2.15) were solved as follows:

$$f(r) = \left(1 - \left(\frac{m}{r}\right)^{10/3}\right)^{-1/2}, \qquad g(r)^2 = \tfrac{9}{20} r^2, \qquad h(r) = -\tfrac{3}{10} r f(r)^{-1}. \tag{2.17}$$

We note that $f(r)$ satisfies the equation

$$r f'(r) = \tfrac{5}{3} f (1 - f^2). \tag{2.18}$$

In eight dimensions the spin connection ω is $SO(8)$ valued. Let Γ_{ab} be a generator of $SO(8)$. The tensor Ψ_{abcd} obeys the identity (B.5) which implies

$$\Psi_{abpq} \Psi^{cdpq} = 6(\delta_a^c \delta_b^d - \delta_a^d \delta_b^c) - 4\Psi_{ab}^{cd}. \tag{2.19}$$

[1] Note again the change of the relative sign in the Euclidean time.

This identity means that if we regard Ψ_{ab}^{cd} as a linear map D on $SO(8)$ algebra, then the eigenvalues of $(1/2)D$ are 1 and -3. Since $\dim SO(8) = 28$ and D is traceless, we get the eigenspace decomposition:

$$SO(8) = E(1) \oplus E(-3), \qquad \dim E(1) = 21, \tag{2.20}$$

where $E(1)$ coincides with $Spin(7)$ subgroup of $SO(8)$ [14]. The orthogonal projection operator to each eigenspace is

$$P_1 = \tfrac{3}{4}\left(1 + \tfrac{1}{6}D\right), \qquad P_{-3} = \tfrac{1}{4}\left(1 - \tfrac{1}{2}D\right). \tag{2.21}$$

We obtain the following generator of $Spin(7)$

$$G_{ab} = \tfrac{3}{4}\left(\Gamma_{ab} + \tfrac{1}{6}\Psi_{abcd}\Gamma^{cd}\right), \tag{2.22}$$

which satisfies the constraint

$$G_{ab} - \tfrac{1}{2}\Psi_{abcd}G^{cd} = 0. \tag{2.23}$$

The algebra $SO(8)$ has four mutually commuting $SU(2)$ subalgebras with the following generators:

$$S^i = \left(\Gamma^{8i} + \tfrac{1}{2}\epsilon_{ijk}\Gamma^{jk}\right), \qquad T^i = \left(\Gamma^{8i} - \tfrac{1}{2}\epsilon_{ijk}\Gamma^{jk}\right), \tag{2.24}$$

$$U^i = \left(\Gamma^{7\hat{i}} + \tfrac{1}{2}\epsilon_{ijk}\Gamma^{\hat{j}\hat{k}}\right), \qquad V^i = \left(\Gamma^{7\hat{i}} - \tfrac{1}{2}\epsilon_{ijk}\Gamma^{\hat{j}\hat{k}}\right). \tag{2.25}$$

By comparing the Dynkin diagrams of $SO(8)$ and $Spin(7)$, we see that $Spin(7)$ is obtained by identifying two of $SU(2)$ subalgebras which correspond to the outer nodes of $SO(8)$ diagram. Indeed the constraint (2.23) implies an identification of $S^i \equiv U^i$, i.e., $SU(2)$ in the base direction and that along the fiber (the normal direction) are identified.[2] Since $\omega_{ab}\Gamma^{ab} = \omega_{ab}(P_1 + P_{-3})\Gamma^{ab} = \omega_{ab}G^{ab} + P_{-3}\omega_{ab}\Gamma_{ab}$, we can regard an $SO(8)$ valued connection with the octonionic self-duality $P_{-3}\omega_{ab} = 0$ as $Spin(7)$ valued. Thus the self-duality (2.16) of the spin connection imposed in [12] is just the requirement that ω is $Spin(7)$ valued. In fact we can prove that the Cayley four-form

$$\Omega = \tfrac{1}{4!}\Psi_{abcd}e^a \wedge e^b \wedge e^c \wedge e^d, \tag{2.26}$$

is closed, if the spin connection satisfies the self-duality (2.16). The closedness of the Cayley four-form is equivalent to the condition of $Spin(7)$ holonomy [16,17]. The total space of the quaternionic line bundle on S^4 with the metric given by (2.17) is a manifold of $Spin(7)$ holonomy.

3. Ansatz for *Spin*(7) connection

Now we consider the octonionic instanton on a $Spin(7)$ bundle over the GPP–BFK metric (octonionic gravitational instanton), which we would like to propose as a higher dimensional

[2] This is definitely related to the topological twist on the world volume of SUSY cycles [15].

generalization of $SU(2)$ instanton on the Eguchi-Hanson metric (four-dimensional gravitational instanton). In the four-dimensional case, the spin connection of the Eguchi-Hanson metric satisfies the self-duality condition with respect to internal $SU(2)$ indices and this implies the curvature two form satisfies the self-duality on space–time indices as well as internal ones. Thus we can obtain an example of Yang–Mills instanton by the standard embedding of the spin connection into the Yang–Mills connection. The same thing takes place for *Spin*(7) case in eight dimensions due to the basic identity for the structure constants of octonions. (For $SU(2)$ case in the above the corresponding object is ϵ_{ijk}; the structure constants of the quaternions.) For the standard embedding of *Spin*(7) connections, see Appendix B.

We now try to find more general *Spin*(7) instantons other than the one obtained by the standard embedding. Motivated by the form of spin connection of GPP–BFK metric, we take the following ansatz for *Spin*(7) connection:

$$A_{ij} = -\left(\frac{1}{g\sin\mu}e^k + Ye^{\hat{k}}\right)\epsilon_{ijk}, \quad A_{\hat{i}\hat{j}} = -\left((X+Y)e^{\hat{k}} + \frac{\cot(\mu/2)}{g}e^k\right)\epsilon_{ijk}, \tag{3.1}$$

$$A_{i\hat{j}} = Z\left(\epsilon_{ijk}e^k + \delta_{ij}e^8\right), \qquad A_{8i} = -\frac{\cot\mu}{g}e^i - Ye^{\hat{i}}, \tag{3.2}$$

$$A_{8\hat{i}} = -Ze^i, \qquad A_{87} = -(3Z)e^8, \tag{3.3}$$

$$A_{7i} = (3Z)e^i, \qquad A_{7\hat{i}} = (X-Y)e^{\hat{i}}, \tag{3.4}$$

where $X(r)$, $Y(r)$, $Z(r)$ are unknown functions of the radial co-ordinate r. For the spin connection we have

$$X(r) = \frac{1}{2h(r)} - \frac{h(r)}{2g(r)^2}, \qquad Y(r) = Z(r) = \frac{h(r)}{2g(r)^2}. \tag{3.5}$$

Since this ansatz satisfies the octonionic self-duality

$$A_{ab} = \tfrac{1}{2}\Psi_{abcd}A^{cd}, \tag{3.6}$$

with respect to the Lie algebra indices, we can consistently regard the connection $A = (1/4)G^{ab}A_{ab}$ as *Spin*(7) valued. We also note that the vielbein of radial direction e^7 does not appear, hence if we think of the radial co-ordinate as a “time”, we are in the temporal gauge.

Due to the identity obeyed by the component of the calibration four-form or the structure constant of octonions, the curvature $F = \mathrm{d}A + A\wedge A$ satisfies the same self-duality as the connection and hence it is also *Spin*(7) valued (i.e., we can compute the curvature as if A was $SO(8)$ connection) We obtain the following curvature:

$$F_{ij} = -\frac{1}{f}\left(Y' + \frac{h'}{h}Y\right)e^7\wedge e^{\hat{k}}\epsilon_{ijk} + \left(\frac{1}{g^2} - \frac{h}{g^2}Y - 10Z^2\right)e^i\wedge e^j + \left(\frac{1}{h}Y - 2Y^2\right)e^{\hat{i}}\wedge e^{\hat{j}} + \left(-\frac{h}{g^2}Y + 2Z^2\right)e^8\wedge e^k\epsilon_{ijk}, \tag{3.7}$$

$$F_{\hat{i}\hat{j}} = -\frac{1}{f}\left(X' + Y' + \frac{h'}{h}(X+Y)\right) e^7 \wedge e^{\hat{k}}\epsilon_{ijk} + \left(\frac{1}{g^2} - \frac{h}{g^2}(X+Y) - 2Z^2\right) e^i \wedge e^j + \left(\frac{1}{h}(X+Y) - 2X^2 - 2Y^2\right) e^{\hat{i}} \wedge e^{\hat{j}} + \left(\frac{1}{g^2} - \frac{h}{g^2}(X+Y) - 2Z^2\right) e^8 \wedge e^k \epsilon_{ijk}, \tag{3.8}$$

$$F_{i\hat{i}} = \frac{1}{f}\left(Z' + \frac{g'}{g}Z\right) e^7 \wedge e^8 + Z(4Y-3X)e^i \wedge e^{\hat{i}} + XZ\sum_{k\neq i} e^k \wedge e^{\hat{k}}, \tag{3.9}$$

$$F_{i\hat{j}} = \frac{1}{f}\left(Z' + \frac{g'}{g}Z\right) e^7 \wedge e^k \epsilon_{ijk} + Z(4Y-3X)e^i \wedge e^{\hat{j}} - XZe^j \wedge e^{\hat{i}} - XZe^8 \wedge e^{\hat{k}}\epsilon_{ijk} \quad (i \neq j) \tag{3.10}$$

$$F_{8i} = -\frac{1}{f}\left(Y' + \frac{h'}{h}Y\right) e^7 \wedge e^{\hat{i}} + \left(\frac{1}{g^2} - \frac{h}{g^2}Y - 10Z^2\right) e^8 \wedge e^i + \left(-\frac{h}{2g^2}Y + Z^2\right)\epsilon_{ijk}\, e^j \wedge e^k + \left(\frac{1}{2h}Y - Y^2\right)\epsilon_{ijk}\, e^{\hat{j}} \wedge e^{\hat{k}}, \tag{3.11}$$

$$F_{8\hat{i}} = -\frac{1}{f}\left(Z' + \frac{g'}{g}Z\right) e^7 \wedge e^i + Z(4Y-3X)e^8 \wedge e^{\hat{i}} - XZ\epsilon_{ijk}\, e^j \wedge e^{\hat{k}}, \tag{3.12}$$

$$F_{87} = -\frac{3}{f}\left(Z' + \frac{g'}{g}Z\right) e^7 \wedge e^8 + Z(X-4Y)\sum_k e^k \wedge e^{\hat{k}}, \tag{3.13}$$

$$F_{7i} = \frac{3}{f}\left(Z' + \frac{g'}{g}Z\right) e^7 \wedge e^i + Z(4Y-X)\epsilon_{ijk}\, e^j \wedge e^{\hat{k}} - Z(4Y-X)e^8 \wedge e^{\hat{i}}, \tag{3.14}$$

$$F_{7\hat{i}} = \frac{1}{f}\left(X' - Y' + \frac{h'}{h}(X-Y)\right) e^7 \wedge e^{\hat{i}} + \left(\frac{h}{2g^2}(X-Y) - 3Z^2\right)\epsilon_{ijk}\, e^j \wedge e^k + \left(-\frac{1}{2h}(X-Y) + X^2 - Y^2\right)\epsilon_{ijk}\, e^{\hat{j}} \wedge e^{\hat{k}} + \left(\frac{h}{g^2}(X-Y) - 6Z^2\right) e^8 \wedge e^i. \tag{3.15}$$

In the case of the Riemannian curvature which comes from the metric the symmetry of four indices implies that the octonionic self-duality as a space–time two form follows from that of $SO(8)$ indices. However, for the Yang–Mills curvature the octonionic instanton equation [9,18,19]:

$$*F = \Omega \wedge F, \tag{3.16}$$

is independent of the self-duality of F with respect to the Lie algebra indices which only implies that it is $Spin(7)$ valued. Thanks to the cyclic symmetry of $SU(2)$ indices in our ansatz, the octonionic self-duality of the curvature two-form is greatly reduced. We find the following three independent conditions:

$$\frac{1}{f}X' + L\cdot X + 2X^2 - \left(\frac{1}{h}+\frac{2h}{g^2}\right)X + \frac{1}{g^2} + 4Z^2 = 0,$$
$$\frac{1}{f}Y' + L\cdot Y + 2Y^2 - \left(\frac{1}{h}+\frac{2h}{g^2}\right)Y + \frac{1}{g^2} - 8Z^2 = 0,$$
$$\frac{1}{f}Z' + K\cdot Z + (X-4Y)Z = 0. \tag{3.17}$$

4. Reduced solutions

Using the explicit form of the functions g and h in (2.17) and making the following change of variables which is convenient for this special background of gravitational instanton

$$X = \frac{1}{rf}\left(\xi_X - \frac{5}{3}f^2\right), \tag{4.1}$$

$$Y = \frac{1}{rf}\left(\xi_Y - \frac{5}{3}f^2\right), \tag{4.2}$$

$$Z = \frac{1}{rf}\xi_Z, \tag{4.3}$$

we obtain the following system of first order ordinary differential equations:

$$r\frac{\mathrm{d}}{\mathrm{d}r}\xi_X + 2\xi_X(\xi_X - 1) + 4\xi_Z^2 = 0, \tag{4.4}$$

$$r\frac{\mathrm{d}}{\mathrm{d}r}\xi_Y + 2\xi_Y(\xi_Y - 1) - 8\xi_Z^2 = 0, \tag{4.5}$$

$$r\frac{\mathrm{d}}{\mathrm{d}r}\xi_Z + (\xi_X - 4\xi_Y + 5)\xi_Z + \frac{20}{3}(f^2-1)\xi_Z = 0. \tag{4.6}$$

Before considering solutions to the differential equations (4.4)–(4.6), let us first look at the asymptotic behavior at infinity. In the region of the large radial co-ordinate r neglecting the last term in (4.6) (or putting $f=1$), we obtain gradient flow equations on $\mathbf{R}^3 = \{\xi = (\xi_X, \xi_Y, \xi_Z)\}$ with a metric $\eta_{AB} = \mathrm{diag}((1/2), 1, 4)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi^A = -\eta^{AB}\frac{\partial U(\xi)}{\partial \xi^B}, \tag{4.7}$$

where $t = \ln r$ and the potential function U is given by

$$U = \tfrac{1}{3}\xi_X^3 - \tfrac{1}{2}\xi_X^2 + \tfrac{2}{3}\xi_Y^3 - \xi_Y^2 + 2\xi_Z^2(\xi_X - 4\xi_Y + 5). \tag{4.8}$$

By counting the number of negative eigenvalues of the Hessian

Table 1
Critical points of the potential function U and their Morse indices

Critical point	Critical values	Index
P_1	(1,1,0)	0
P_2	(1,0,0)	1
P_3	(0,1,0)	1
P_4	(0,0,0)	2
$P_5^{\pm}$	$(1/3, 4/3, \pm 1/3)$	2
$P_6^{\pm}$	$\left(5/11, 15/11, \pm\sqrt{15}/11\right)$	1

$$H_{AB} = \frac{\partial^2 U}{\partial \xi_A \partial \xi_B} = \begin{pmatrix} 2\xi_X - 1 & 0 & 4\xi_Z \\ 0 & 4\xi_Y - 2 & -16\xi_Z \\ 4\xi_Z & -16\xi_Z & 4(\xi_X - 4\xi_Y + 5) \end{pmatrix}, \tag{4.9}$$

at the critical points of U, we find the list of the critical points and the Morse indices (see Table 1). We can use one of these critical points as a boundary condition of the octonionic Yang–Mills instantons and classify the solutions of (4.4)–(4.6) according to

$$\mathit{Sol}(P_i) := \{\text{a family of solutions approaching to } P_i \text{ for } r \to \infty\}. \tag{4.10}$$

If $\xi_Z = 0$, we can neglect the last term of (4.6) in the whole region and we can find a general solution:

$$\xi_X = \frac{1}{1-(a/r^2)}, \qquad \xi_Y = \frac{1}{1-(b/r^2)}, \tag{4.11}$$

which belongs to the class $\mathit{Sol}(P_1)$ for finite parameters (a, b). When we take the limits of parameters (a, ∞), (∞, b) and (∞, ∞), the limiting solutions belong to $\mathit{Sol}(P_2)$, $\mathit{Sol}(P_3)$ and $\mathit{Sol}(P_4)$, respectively. When $\xi_Z \neq 0$ it is difficult to find general solutions, which will be discussed in more detail in Section 5, but we here present a special solution in $\mathit{Sol}(P_5^-)$ corresponding to the spin connection

$$\xi_X = \tfrac{1}{3}, \qquad \xi_Y = -\tfrac{1}{3} + \tfrac{5}{3} f^2, \qquad \xi_Z = -\tfrac{1}{3}. \tag{4.12}$$

Note that provided a solution $(\xi_X, \xi_Y, \xi_Z) \in \mathit{Sol}(P_{5,6}^+)$ there always exists a solution $(\xi_X, \xi_Y, -\xi_Z) \in \mathit{Sol}(P_{5,6}^-)$ by the symmetry of equations.

The spin connection (4.12) yields a $\mathit{Spin}(7)$ connection, while as we will see shortly the connection of (4.11) takes the value in a sub-Lie algebra of $\mathit{Spin}(7)$. In this sense (4.11) is a reduced solution. Let us calculate the curvature two-form $F = (1/4)G^{ab}F_{ab}$ by substituting (4.1) and (4.2) and $Z = 0$ into (3.7)–(3.15). Defining the sub-generators of G^{ab} by

$$S^i = \tfrac{1}{4}\left(-G^{7\hat{i}} + \tfrac{1}{2}\epsilon_{ijk}G^{\hat{j}\hat{k}}\right), \tag{4.13}$$

$$T^i = \tfrac{1}{2}\left(G^{8i} + \tfrac{1}{2}\epsilon_{ijk}G^{jk}\right) = \tfrac{1}{2}\left(G^{7\hat{i}} + \tfrac{1}{2}\epsilon_{ijk}G^{\hat{j}\hat{k}}\right), \tag{4.14}$$

$$U^i = \tfrac{1}{4}\left(-G^{8i} + \tfrac{1}{2}\epsilon_{ijk}G^{jk}\right), \tag{4.15}$$

we can rewrite the curvature into the following form:

$$F = S^i F_i(\xi_X) + T^i F_i(\xi_Y) + U^i C_i, \tag{4.16}$$

where the two-forms $F_i(\xi_A)$ $(A = X, Y)$ and C_i are given by

$$F_i(\xi_A) = f_A \left(e^8 \wedge e^i + \frac{1}{2}\epsilon_{ijk} e^j \wedge e^k \right) + \frac{g_A}{2}\epsilon_{ijk} e^{\hat{j}} \wedge e^{\hat{k}} + h_A e^7 \wedge e^{\hat{i}}, \tag{4.17}$$

with

$$f_A = \frac{4\xi_A}{3r^2 f^2}, \qquad g_A = -\frac{4\xi_A}{r^2 f^2}\left(\xi_A - \frac{5}{3}f^2\right),$$
$$h_A = \frac{4\xi_A}{r^2 f^2}\left(\xi_A + \frac{2}{3} - \frac{5}{3}f^2\right), \tag{4.18}$$

and

$$C_i = \frac{20}{9r^2}\left(-e^8 \wedge e^i + \frac{1}{2}\epsilon_{ijk} e^j \wedge e^k\right). \tag{4.19}$$

It is easy to see that S^i, T^i and U^i are mutually commuting $SU(2)$ generators which satisfy the relations

$$[S^i, S^j] = -\epsilon_{ijk} S^k, \qquad [T^i, T^j] = -\epsilon_{ijk} T^k, \qquad [U^i, U^j] = -\epsilon_{ijk} U^k. \tag{4.20}$$

Thus the solution (4.11) describes the $SU(2)^3$ octonionic Yang–Mills instanton. When the parameters (a, b) take the special values, some curvature components vanish:

$$F_i(\xi_Y) = 0 \quad \text{for} \ (a, \infty), \tag{4.21}$$

$$F_i(\xi_X) = 0 \quad \text{for} \ (\infty, b), \tag{4.22}$$

$$F_i(\xi_X) = F_i(\xi_Y) = 0 \quad \text{for} \ (\infty, \infty). \tag{4.23}$$

The gauge group is then further reduced to $SU(2)^2$ and $SU(2)$ in the case of (4.21) or (4.22) and (4.23), respectively.

We now evaluate the Chern forms characterizing the topological nature of the solution (4.11). The relevant closed forms are given by

$$c_2 = \frac{1}{8\pi^2}\,\mathrm{Tr}\, F \wedge F, \tag{4.24}$$

$$c_4 = \frac{1}{128\pi^4}\left((\mathrm{Tr}\, F \wedge F)(\mathrm{Tr}\, F \wedge F) - 2\mathrm{Tr}\, F \wedge F \wedge F \wedge F\right), \tag{4.25}$$

where Tr refers to the adjoint representation of $SU(2)^3$. Since c_2 is a four-form, it must be integrated over a four-dimensional hypersurface in the quaternionic line bundle. A natural choice is the fiber $\mathbf{R}^4$, which is specified by the orthonormal frame $\{e^4, e^5, e^6, e^7\}$. Using (4.11),(4.17) and (4.19), we obtain the formula

$$\int c_2 = \left(\frac{3}{10}\right)^3 \frac{3!}{\pi^2} vol_{SU(2)} \sum_{A=X,Y} I_A, \tag{4.26}$$

where

$$I_A = -\frac{1}{4}\int_m^\infty \mathrm{d}r\, r^3 f^{-2} g_A h_A = \frac{1}{3}\int_m^\infty \mathrm{d}r \frac{\mathrm{d}}{\mathrm{d}r}\left(5\xi_A^2 f^{-4} - 2\xi_A^3 f^{-6}\right) \tag{4.27}$$

in the parameter region $a, b < m^2$. Since $f^{-1} = 0$ at $r = m$ and $\xi_X = \xi_Y = f = 1$ for $r \to \infty$, we have $I_X = I_Y = 1$. For c_4 the integration is to be evaluated over the total space of the quaternionic line bundle. A similar calculation yields

$$\int c_4 = \left(\frac{3}{10}\right)^5 \frac{4}{\pi^4} vol_{S^7} \left(25 \sum_{A=X,Y} I_A - I_{XY}\right), \tag{4.28}$$

where

$$\begin{aligned} I_{XY} &= -\left(\frac{3}{4}\right)^3 \sum_{A\neq B}\int_m^\infty \mathrm{d}r\, r^7 f^{-2}\left(3f_A^2 g_B h_B + 2 f_A f_B g_A g_B\right) \\ &= \int_m^\infty \mathrm{d}r \frac{\mathrm{d}}{\mathrm{d}r}\left(25 f^{-8}\xi_X^2\xi_Y^2 - 6 f^{-10}\xi_X^2\xi_Y^2(\xi_X + \xi_Y)\right), \end{aligned} \tag{4.29}$$

and $I_{XY} = 13$ for $a, b < m^2$.

5. Asymptotic expansion

As we have mentioned it is difficult to obtain a solution with $\xi_Z \neq 0$ in a closed form. The only exceptional solution is given by the spin connection regarded as a *Spin*(7) gauge field. To investigate the existence of more general solutions let us look for a solution in a form of formal power series. We first consider the exponent of the power series determined by the gradient flow equation. Linearizing (4.7) around the critical point P_i, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\xi}_A = -\tilde{H}_{AB}\tilde{\xi}_B, \tag{5.1}$$

where $\tilde{H}_{AB}$ represents the 3×3 matrix $\eta^{AC} H_{CB}$ evaluated at P_i. To be specific, let $\lambda_1, \lambda_2, \ldots$ be the positive eigenvalues of $\tilde{H}$ and $\xi_A(P_i)$ the critical values. Then, the gradient flow approaching to P_i may be expanded in the form

$$\xi_A = \xi_A(P_i) + a_1^A\left(\frac{a}{r}\right)^{\lambda_1} + a_2^A\left(\frac{b}{r}\right)^{\lambda_2} + \cdots \tag{5.2}$$

for a large radial co-ordinate. Here coefficients $a_1^A, a_2^A, \ldots$ are determined by the linearized gradient flow equation (5.1) and $a, b, \ldots$ are $3 - index(P_i)$ free parameters. Recall that the gradient flow gives an approximate solution, and hence the expansion (5.2) must be corrected by the exact octonionic Yang–Mills equation. The correction comes from the last term $(f^2 - 1)\xi_Z$ in (4.6), which is also expanded by using

$$f^2 = \sum_{n=0}^{\infty}\left(\frac{m}{r}\right)^{(10n)/3}. \tag{5.3}$$

Now we make an ansatz of the formal power series which takes into account the asymptotic behavior described above:

$$\xi_A \approx \xi_A(P_i) + \sum_{k=0}^{\infty}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \cdots (w^k z_1^{n_1} z_2^{n_2} \cdots)\, a^A_{kn_1n_2\cdots}, \quad k + n_1 + n_2 + \cdots \neq 0, \tag{5.4}$$

where $w = (m/r)^{10/3}$, $z_1 = (a/r)^{\lambda_1}$, $z_2 = (b/r)^{\lambda_2}, \ldots$. This series will provide the asymptotic expansion of the octonionic Yang–Mills instanton in $Sol(P_i)$ with totally $4 - index(P_i)$ moduli parameters, if the parameter m of the background metric is treated as a moduli. In the following we illustrate this asymptotic expansion by concentrating on the cases $Sol(P_5^-)$ and $Sol(P_1)$. Other cases can be analyzed in the same manner.

(a) $Sol(P_5^-)$. The matrix $\tilde{H}$ has one positive eigenvalue $\lambda = (1/3)(7 + \sqrt{57})$, so that the solution is written as the double power series:

$$\xi_A \approx \xi_A(P_5^-) + \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} w^n z^k a^A_{nk}, \quad n + k \neq 0, \tag{5.5}$$

where $w = (m/r)^{10/3}$, $z = (a/r)^{\lambda}$ and we have one free (moduli) parameter a in addition to the background metric moduli m. When we take $a^X_{01} = a^Y_{01} = a^Z_{01} = 0$ as an initial condition of the recursion relation, we find a solution

$$a^Y_{n0} = \tfrac{5}{3} \quad (\forall n \geq 1), \qquad \text{others} = 0, \tag{5.6}$$

which recovers the spin connection (4.12). More general solution with the additional moduli parameter a is obtained by using the series

$$\xi_X \approx \frac{1}{3} + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} w^n z^k a_{nk}, \tag{5.7}$$

$$\xi_Y \approx -\frac{1}{3} + \frac{5}{3} f^2 + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} w^n z^k b_{nk}, \tag{5.8}$$

$$\xi_Z \approx -\frac{1}{3} + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} w^n z^k c_{nk}, \tag{5.9}$$

where the first coefficients are $a_{01} = -(1/3)(9 - \sqrt{57})$, $b_{01} = (1/3)(3 + \sqrt{57})$ and $c_{01} = 1$. The higher coefficients are uniquely determined by the recursion formulae:

$$\left(\frac{10}{3}n + \lambda k + \frac{2}{3}\right) a_{nk} + \frac{8}{3} c_{nk} - 2\sum_{p=0}^{n}\sum_{q=1}^{k-1} \left(a_{pq} a_{n-p,k-q} + 2c_{pq} c_{n-p,k-q}\right) = 0, \tag{5.10}$$

$$\left(\frac{10}{3}n+\lambda k-\frac{10}{3}\right)b_{nk}-\frac{16}{3}c_{nk}-\frac{20}{3}\sum_{p=0}^{n-1}b_{pk} -2\sum_{p=0}^{n}\sum_{q=1}^{k-1}\left(b_{pq}b_{n-p,k-q}-4c_{pq}c_{n-p,k-q}\right)=0, \tag{5.11}$$

$$\frac{1}{3}a_{nk}-\frac{4}{3}b_{nk}+\left(\frac{10}{3}n+\lambda k\right)c_{nk}-\sum_{p=0}^{n}\sum_{q=1}^{k-1}c_{pq}\left(a_{n-p,k-q}-4b_{n-p,k-q}\right)=0. \tag{5.12}$$

We leave the issue of convergence of this formal solution for future research. This is important for a global structure of the moduli space such as a compactification.

(b) *Sol*(P_1). The matrix $\tilde{H}$ has degenerated eigenvalues, i.e., $\lambda_1=\lambda_2=\lambda_3=2$. In order to unambiguously determine the coefficients $a^A_{kn_1n_2n_3}$ of power series, we first perturb the octonionic Yang–Mills equation so that the corresponding $\tilde{H}$ has non-degenerate eigenvalues and after all calculations we take off the perturbation. If we use the metric $\eta_{AB}=\mathrm{diag}((1/2)(1-\epsilon_1),1-\epsilon_2,4)$ with small parameters ϵ_1 and ϵ_2 instead of $\eta_{AB}=\mathrm{diag}(1/2,1,4)$, then the positive eigenvalues of $\tilde{H}$ are $\lambda_1=2(1+\epsilon_1)$, $\lambda_2=2(1+\epsilon_2)$, $\lambda_3=2$. Eqs. (4.4) and (4.5) are replaced by

$$r\frac{\mathrm{d}}{\mathrm{d}r}\xi_X+\lambda_1\xi_X(\xi_X-1)+2\lambda_1\xi_Z^2=0, \tag{5.13}$$

$$r\frac{\mathrm{d}}{\mathrm{d}r}\xi_Y+\lambda_2\xi_Y(\xi_Y-1)-4\lambda_2\xi_Z^2=0, \tag{5.14}$$

while (4.6) for the Z-component remains unchanged. After taking the limit $\epsilon_i\to 0$ $(i=1,2)$, the solution in a form of power series is given by (5.4) with $w=(m/r)^{10/3}$, $z_1=(a/r)^2$, $z_2=(b/r)^2$ and $z_3=(c/r)^2$. The explicit lower terms are as follows:

$$\xi_X\approx\frac{1}{1-z_1}+\sum_{n=1}^{\infty}z_3^n a_n+\cdots, \tag{5.15}$$

$$\xi_Y\approx\frac{1}{1-z_2}+\sum_{n=1}^{\infty}z_3^n b_n+\cdots, \tag{5.16}$$

$$\xi_Z\approx\sum_{n=1}^{\infty}z_3^n c_n+\cdots, \tag{5.17}$$

where the coefficients are determined by the recursion formulae

$$a_n=\frac{1}{n-1}\sum_{k=1}^{n-1}(a_k a_{n-k}+2c_k c_{n-k}), \tag{5.18}$$

$$b_n = \frac{1}{n-1}\sum_{k=1}^{n-1}(a_k a_{n-k} - 4c_k c_{n-k}), \tag{5.19}$$

$$c_n = \frac{1}{2(n-1)}\sum_{k=1}^{n-1}c_k(a_{n-k} - 4b_{n-k}), \tag{5.20}$$

with the first terms $a_1 = b_1 = 0$ and $c_1 = 1$. If we impose $c = 0$, the above expansion reproduces the exact solution (4.11).

6. Discussion

Firstly we remark the relationship between the asymptotic flow (4.7) and the Chern–Simons theory over the squashed seven sphere $\hat{S}^7$. A similar relation in various dimensions has been discussed in [20]. The gravitational instanton metric (2.1) has the conformally product form in a large radial co-ordinate region,

$$\mathrm{d}s^2 \to \mathrm{d}r^2 + \frac{9}{20}r^2\,\mathrm{d}s^2_{\hat{S}^7}, \tag{6.1}$$

where $\mathrm{d}s^2_{\hat{S}^7}$ is the $Sp(2)\cdot Sp(1)$-invariant metric on $\hat{S}^7$:

$$\mathrm{d}s^2_{\hat{S}^7} = \mathrm{d}\mu^2 + \tfrac{1}{4}\sin^2\mu\cdot\Sigma_i^2 + \tfrac{1}{5}(\sigma_i - A_i)^2. \tag{6.2}$$

Now (4.7) may be regarded as the gradient flow of the Chern–Simons theory on $\hat{S}^7$ as follows. Let us define the action by

$$CS[A] = \int_{\hat{S}^7}\hat{\Psi}\wedge\mathrm{Tr}\left(A\wedge\mathrm{d}A + \tfrac{2}{3}A\wedge A\wedge A\right), \tag{6.3}$$

where A stands for $Spin(7)$ connection and $\hat{\Psi}$ a closed four-form on $\hat{S}^7$ induced by the calibration four-form on the total space with $Spin(7)$ holonomy. It is explicitly written as

$$\begin{aligned}\hat{\Psi} = {}&\theta^1\wedge\theta^2\wedge\theta^3\wedge\theta^8 - \theta^1\wedge\theta^{\hat{2}}\wedge\theta^{\hat{3}}\wedge\theta^8 + \theta^2\wedge\theta^{\hat{1}}\wedge\theta^{\hat{3}}\wedge\theta^8\\ &-\theta^3\wedge\theta^{\hat{1}}\wedge\theta^{\hat{2}}\wedge\theta^8 + \theta^1\wedge\theta^2\wedge\theta^{\hat{1}}\wedge\theta^{\hat{2}}\\ &+\theta^2\wedge\theta^3\wedge\theta^{\hat{2}}\wedge\theta^{\hat{3}} + \theta^1\wedge\theta^3\wedge\theta^{\hat{1}}\wedge\theta^{\hat{3}},\end{aligned} \tag{6.4}$$

using the orthonormal frame of the metric (6.2)

$$\theta^i = \tfrac{1}{2}\sin\mu\cdot\Sigma_i, \qquad \theta^{\hat{i}} = \tfrac{1}{\sqrt{5}}(\sigma_i - A_i), \qquad \theta^8 = d\mu. \tag{6.5}$$

The gradient flow of the functional (6.3) obeys a differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}A = -*(\hat{\Psi}\wedge F), \tag{6.6}$$

which reproduces the flow (4.7) if we use the ansatz (3.1)–(3.4) and (4.1)–(4.3) with $f = 1$. The critical points of $CS[A]$, i.e., the solutions of $\hat{\Psi}\wedge F = 0$, then reduce to those of the potential function U, which does not mean flat-connections as in the three-dimensional Chern–Simons theory.

The energy–momentum tensor of the Yang–Mills field is

$$T^{\rm YM}_{\mu\nu} = {\rm Tr}\left(F_{\mu\lambda}F^{\lambda}_{\nu}\right) - \tfrac{1}{4}g_{\mu\nu}{\rm Tr}\left(F_{\lambda\kappa}F^{\lambda\kappa}\right). \qquad (6.7)$$

For the octonionic instantons the energy–momentum tensor $T_{\mu\nu}$ does not vanish in general and this presents a sharp contrast with the case of four-dimensional instantons. We can prove that (anti-) self-duality of the curvature implies $T^{\rm YM} = 0$ using a property of the totally anti-symmetric tensor $\epsilon_{\mu\nu\rho\sigma}$ in four dimensions. Hence a four-dimensional instanton does not disturb the Ricci flatness of the background metric. However the octonionic self-duality is not sufficient for leading a vanishing energy–momentum tensor and this causes an issue of the back reaction of matter to gravity. Non-vanishing total energy–momentum tensor is inconsistent with the Ricci flatness of the space–time. Actually a similar issue is encountered, if we embed a four-dimensional instanton in higher dimensions. (Note that this gives a special case of higher dimensional instanton.) In such a case, though the tangent components of $T_{\mu\nu}$ along the four-dimensional submanifold on which the instanton lives vanish as we argued above, the instanton produces a non-vanishing contribution to the $(d-4)$-dimensional normal components of the energy–momentum tensor. One of the ways to resolve this problem is to embed the solution to a consistent background of superstrings. Based on a four-dimensional instanton, one can construct several five brane solutions which has a dilaton ϕ and an anti-symmetric tensor field $H = {\rm d}B$ in addition to the Yang–Mills field A [21–24]. In these solutions we can think of four-dimensional instanton as a source of configuration in the transverse space. Due to a coupling

$${\rm d}H = {\rm Tr}(R \wedge R - F \wedge F), \qquad (6.8)$$

there are contributions from ϕ and B which balances the total energy–momentum tensor. This should be so, since this provides a consistent background for superstrings. From this example we believe our solutions should be promoted to a consistent background of supermembrane or eleven-dimensional super gravity [12,25,26].

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Appendix A. Quaternionic Kähler manifold

The holonomy group of a connected and orientable $4n$ dimensional Riemannian manifold is a subgroup of $SO(4n)$. If the holonomy group is reduced to $Sp(n) \cdot Sp(1) \cong Sp(n) \times$

$(Sp(1)/\mathbf{Z}_2)$, the manifold is called quaternionic Kähler [17]. (Caution: a quaternionic Kähler manifold is not necessarily Kähler!) The quaternionic Kähler manifold is known as a target space geometry of $N = 2$ supergravity in four dimensions. (If we consider $N = 2$ global SUSY, then $Sp(1)$ part is trivial and the manifold is hyperKähler.) It also provides a natural arena for a higher dimensional generalization of instanton equation. Note that when $n = 1$ the holonomy of quaternionic Kähler manifold is $Sp(1) \times (Sp(1)/\mathbf{Z}_2)$ and hence any four-dimensional orientable manifold is quaternionic Kähler. This may be compared with the fact any two-dimensional orientable manifold is Kähler due to $SO(2) \cong U(1)$.

On the frame bundle of a quaternionic Kähler manifold the gauge transformation among local co-ordinate patches is in $Sp(n) \cdot Sp(1)$. We can define a quaternionic line bundle as an associated bundle to the frame bundle as follows. Each fiber of quaternionic line bundle is the space of quaternions $\mathbf{H}$, i.e., $\mathbf{R}^4$ as a vector space. On quaternions there is a natural action of $Sp(1) \cong SU(2)$. Thus we can define the gauge transformation of the fiber using the $Sp(1)$ part of the holonomy group. That is, $Sp(n)$ part acts trivially on the quaternionic line bundle. This especially implies that quaternionic line bundle on hyperKähler manifold is trivial. It might be helpful to note that this construction is a quaternionic analogue of complex line bundle (or the canonical $U(1)$ bundle) on a Kähler manifold that has $U(n)$ holonomy.

If we take a four-dimensional spin manifold M, each factor of $Sp(1)$ is identified as $Spin(3)$ that defines the spinor bundle $S^{\pm}(M)$. Hence, the complexified spinor bundle on four-dimensional manifold is regarded as a quaternionic line bundle. It is known that the total space of spinor bundle on a four-dimensional manifold possesses a natural $Spin(7)$ structure.

Appendix B. Standard embedding of *Spin*(7) connection

We show that the octonionic self-duality of the spin connection ω with respect to the (local frame) Lie algebra indices implies the octonionic self-duality of the curvature $R = \mathrm{d}\omega + \omega \wedge \omega$ with respect to the space time form indices. Let $\omega_{ab} = -\omega_{ba}$ be the spin connection one-form. In eight dimensions ω is $SO(8)$ valued in general. (We take the Euclidean signature in the following.) But if we impose the octonionic self-duality

$$\omega_{ab} = \tfrac{1}{2}\Psi_{abcd}\omega^{cd}, \tag{B.1}$$

the spin connection ω can be regarded as $Spin(7)$ valued. The totally anti-symmetric tensor Ψ_{abcd} is related to the structure constants C_{abc} of the octonion algebra as follows [27]:

$$\Psi_{abc8} = C_{abc}, \qquad \Psi_{abcd} = \frac{1}{3!}\epsilon_{abcdpqr}C^{pqr}, \tag{B.2}$$

where the duality is taken in seven dimensions. In terms of a local frame (vielbein) e^a_μ we can define a space–time self-dual four-form

$$\Omega = \frac{1}{4!}\Psi_{abcd}e^a \wedge e^b \wedge e^c \wedge e^d. \tag{B.3}$$

On an eight-dimensional manifold of $Spin(7)$ holonomy, the four-form Ψ is closed:

$$\mathrm{d}\Omega = 0, \tag{B.4}$$

A crutial point is the following identity which follows from the property of the octonionic structure constants C_{abc} [14,27],

$$\Psi_{abcd}\Psi^{fghd} = \left(\delta_a^f\delta_b^g - \delta_b^f\delta_a^g\right)\delta_c^h + ((fgh) : cyclic) - \left((\Psi_{ab}^{fg}\delta_c^h + \Psi_{bc}^{fg}\delta_a^h + \Psi_{ca}^{fg}\delta_b^h) + ((fgh) : cyclic)\right). \tag{B.5}$$

Due to the symmetry of the Riemann curvature tensor the self-duality of form indices follows form that of Lie algebra indices. Thus it is enough to show that

$$R_{ab} = \tfrac{1}{2}\Psi_{abcd}R^{cd}, \tag{B.6}$$

where we have suppressed the two-form indices. In the defining relation

$$R_{ab} = \mathrm{d}\omega_{ab} + \omega_{ac}\wedge\omega_{cb}, \tag{B.7}$$

the self-duality of the second term is non-trivial. Using the identity (B.5), we can show the second term is indeed self-dual. Thus the curvature two form constructed from a $Spin(7)$ valued spin connection ω satisfies the octonionic instanton equation.

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